AERO 422: Active Controls for Aerospace Vehicles

Proportional, Integral & Derivative Control Design

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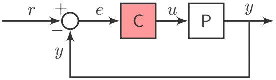
Proportional Control

Proportional Control

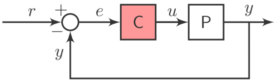


- $C(s) = K$, constant gain

$$u(t) = Ke(t)$$

- Use Routh's criterion to determine range for K
- Verify if the system is stabilizable with $C(s) = K$

Second order system



- $P(s) = \frac{A}{(\tau_1 s+1)(\tau_2 s+1)}, C(s) = K$
- Characteristic equation: zeros of $1 + PC$

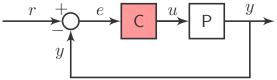
$$\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + AK + 1 = 0$$

- Open loop poles at $-1/\tau_1, -1/\tau_2$
- Closed loop poles at

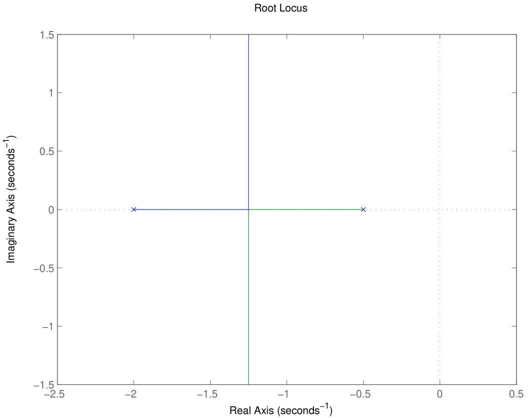
$$p_1 = -\frac{\tau_1 + \tau_2 + \sqrt{{\tau_1}^2 - 2\,\tau_1\,\tau_2 + {\tau_2}^2 - 4\,AK\,\tau_1\,\tau_2}}{2\,\tau_1\,\tau_2}$$

$$p_2 = -\frac{\tau_1 + \tau_2 - \sqrt{{\tau_1}^2 - 2\,\tau_1\,\tau_2 + {\tau_2}^2 - 4\,AK\,\tau_1\,\tau_2}}{2\,\tau_1\,\tau_2}$$

Routh Stability Criterion



$$P(s) = \frac{A}{(\tau_1 s+1)(\tau_2 s+1)}, C(s) = K$$



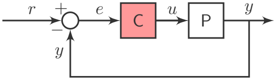
Range for stabilization $K > 0$

$$\begin{array}{c|cc} s^2 & \tau_1\,\tau_2 & AK \\ s^1 & \tau_1 + \tau_2 & 0 \\ s^0 & AK & 0 \end{array}$$

- Poles move towards each other till $K = K^* = \frac{(\tau_1 - \tau_2)^2}{4\tau_1\tau_2 A}$
- $K > K^*$, poles are purely imaginary
- $K > K^*$, $\omega_n \uparrow$, $\zeta \downarrow$

Proportional Control

Steady State Error



System

$$P(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)}, C(s) = K_p$$

Transfer Function

$$G_{yr}(s) := \frac{PC}{1+PC} = \frac{AK_p}{\tau_1\tau_2 s^2 + (\tau_1 + \tau_2)s + AK_p + 1}.$$

Corresponding ODE

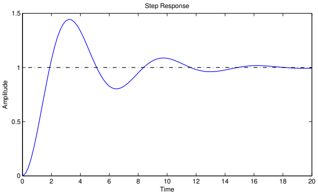
$$\tau_1\tau_2\ddot{y} + (\tau_1 + \tau_2)\dot{y} + (AK_p + 1)y = AK_p r$$

Steady state

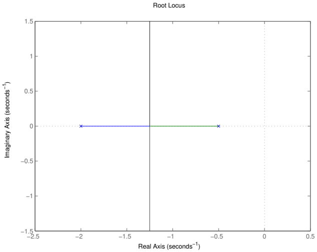
$$\ddot{y} = \dot{y} = 0 \implies y(t) = \frac{AK_p}{1 + AK_p} r(t).$$

Proportional Control

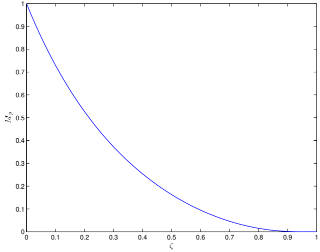
Summary



(a) Step Response



(b) Root Locus



(c) M_p vs ζ

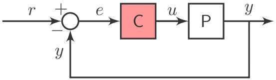
- $K > K^*$, $\omega_n \uparrow$, $\zeta \downarrow$
- $t_r = \frac{1.8}{\omega_n}$, $t_s = \frac{4.6}{\sigma}$

Effect of Proportional Control

- Reduces rise time Good!
- Increases overshoot Bad!
- Large gain $\Rightarrow$ small steady state error
- Amplifies noise and disturbances Bad!
  - Look at frequency characteristics of all the transfer functions (later)

Integral Control

Integral Control

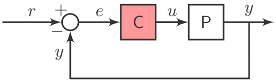


- $C(s) = K_I/s$

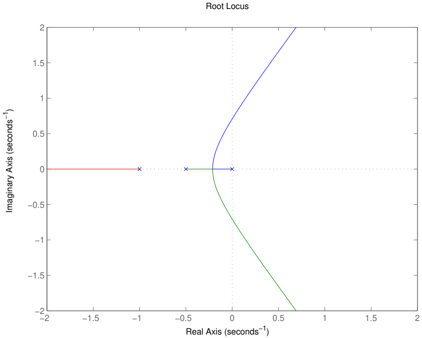
$$u(t) = K_I \int_0^t e(t)$$

- Use Routh's criterion to determine range for K_I
- Verify if the system is stabilizable with $C(s) = K_I/s$

Second Order System



- $P(s) = \frac{A}{(\tau_1 s+1)(\tau_2 s+1)}, C(s) = K_I/s$
- Characteristic equation: $\tau_1\tau_2 s^3 + (\tau_1+\tau_2)s^2 + s + AK_I$

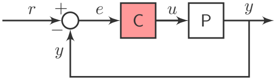


$$
\begin{array}{c|cc}
s^3 & \tau_1\,\tau_2 & 1 \\
s^2 & \tau_1+\tau_2 & AK_I \\
s^1 & \frac{\tau_1+\tau_2-AK_I\,\tau_1\tau_2}{\tau_1+\tau_2} & 0 \\
s^0 & AK_I & 0
\end{array}
$$

$$
0 < K_I < \frac{\tau_1+\tau_2}{A\tau_1\tau_2}
$$

Integral Control

Steady State Error



System

$$P(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)}, C(s) = K_I/s$$

Transfer Function

$$G_{yr}(s) := \frac{PC}{1+PC} = \frac{AK_I}{\tau_1\tau_2 s^3 + (\tau_1+\tau_2)s^2 + AK_I}.$$

Corresponding ODE

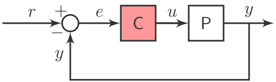
$$\tau_1\tau_2 \dddot{y} + (\tau_1+\tau_2)\ddot{y} + AK_I y = AK_I r$$

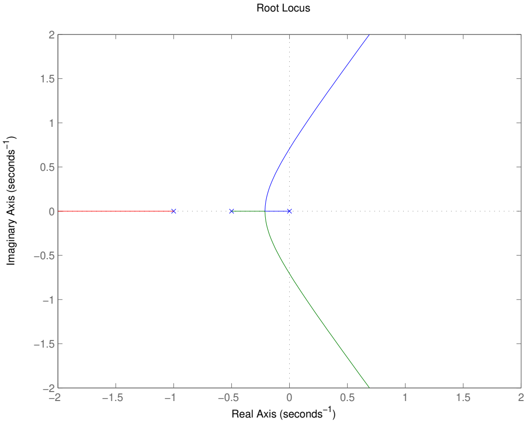
Steady state

$$\dddot{y} = \ddot{y} = 0 \implies y(t) = r(t).$$

Integral Control

Summary

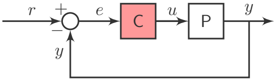




- $K_I \uparrow \implies \zeta \downarrow \implies M_p \uparrow$
- $K_I \uparrow \implies \omega_n \uparrow \implies t_r \downarrow$
- $K_I > \frac{\tau_1+\tau_2}{A\tau_1\tau_2}$ unstable.
- Zero steady state error
- Needs anti windup mechanism for saturated actuators discussed later
- Good disturbance rejection property read book

PI Control

Proportional-Integral Control



System

$$P(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)}, C(s) = K\left(1 + \frac{1}{T_I s}\right)$$

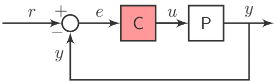
- T_I is called the integral, or reset time
- $1/T_I$ is reset rate, related to speed of response
- $u(t)$ is a mixture of two signals

$$u(t) = K\left(e(t) + \frac{1}{T_I}\int_{t_0}^{t} e(t)d\tau\right).$$

- $u(t) \neq 0$ even when $e(t) = 0$, because of integral action

Proportional-Integral Control

Steady State Error



System

$$P(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)}, C(s) = K\left(1 + \frac{1}{T_I s}\right)$$

Transfer Function

$$G_{yr}(s) := \frac{PC}{1+PC} = \frac{AK(T_I s + 1)}{T_I \tau_1 \tau_2 s^3 + T_I(\tau_1 + \tau_2)s^2 + T_I(1 + AK)s + AK}.$$

Corresponding ODE

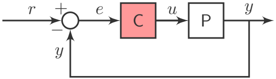
$$(*)\dddot{y} + (*)\ddot{y} + (*)\dot{y} + AKy = AKr + (*)\dot{r}$$

Steady state

$$\dddot{y} = \ddot{y} = \dot{y} = \dot{r} = 0 \implies y(t) = r(t).$$

Proportional-Integral Control

Independent control over two of the three terms



System

$$P(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)}, C(s) = K\left(1 + \frac{1}{T_I s}\right)$$

Transfer Function

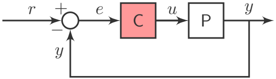
$$G_{yr}(s) := \frac{PC}{1 + PC} = \frac{AK(T_I s + 1)}{T_I \tau_1 \tau_2 s^3 + T_I(\tau_1 + \tau_2)s^2 + T_I(1 + AK)s + AK}.$$

Characteristic Equation

$$T_I \tau_1 \tau_2 s^3 + T_I(\tau_1 + \tau_2)s^2 + T_I(1 + AK)s + AK = 0.$$

Proportional-Integral Control

Two tuning variables



Routh's Table

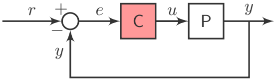
$$
\begin{array}{c|ll}
s^3 & T_I\tau_1\tau_2 & T_I(AK+1) \\
s^2 & T_I(\tau_1+\tau_2) & AK \\
s^1 & T_I + AKT_I - AK\tau_1 + \frac{AK{\tau_1}^2}{\tau_1+\tau_2} & 0 \\
s^0 & AK & 0
\end{array}
$$

Constraints

$$
K > 0, \qquad T_I > \frac{AK}{1+AK}\frac{\tau_1\tau_2}{\tau_1+\tau_2}.
$$

Derivative Control

Derivative Control



- $C(s) = K_D s$

$$u(t) = K_D \dot{e}(t)$$

- Use Routh's criterion to determine range for K_D
- Verify if the system is stabilizable with $C(s) = K_D s$
- Almost never used by itself usually augmented by proportional control
- Derivative control is not causal depends on future values

$$\dot{e} \approx \frac{e(t+\Delta t) - e(t)}{\Delta t}$$

  - Knows the slope $\leftarrow$ known from future values of $e(t)$
  - $e(t) = t, \dot{e} = 1.$
  - $e(t) = t^2, \dot{e} = 2t.$
  - $e(t) = \sin(t), \dot{e} = \cos(t) = \sin(t + \pi/2).$

Approximate Derivative Control

Approximation over Frequency range

- $C(s) = K_D s$

$$u(t) = K_D \dot{e}(t)$$

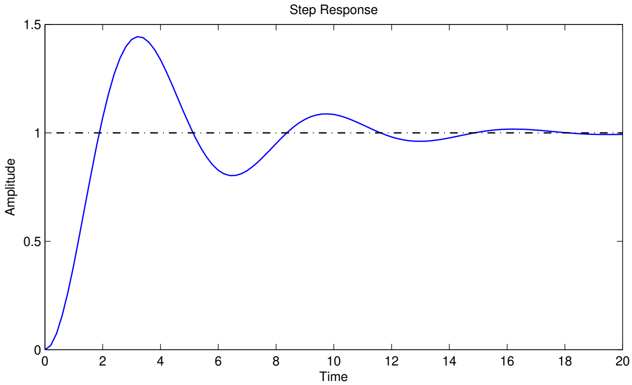
- Not implementable in applications
- Approximate as

$$C(s) = \frac{K_D s}{s/\alpha + 1}, \alpha >> 1 \text{ pole at far left}$$
$$\approx K_D s, \text{ for small } s$$

- What is the effect of this approximation?
- Look at a step response

Approximate Derivative Control

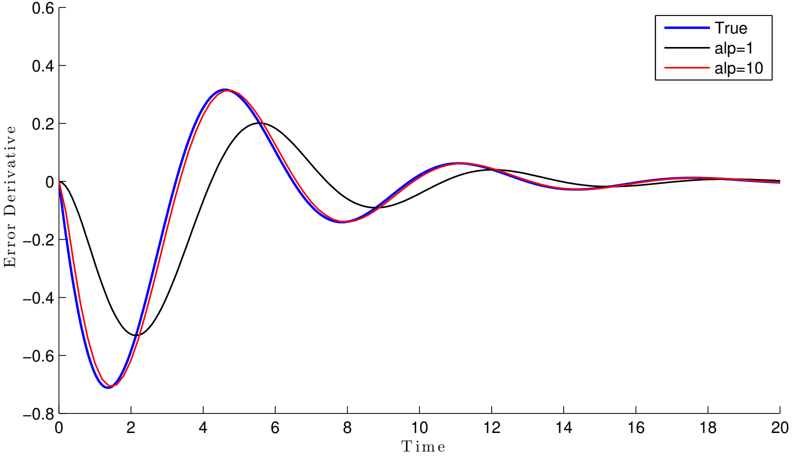
Effect of the Approximation



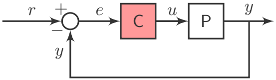
$$e(t) = r(t) - y(t) = 1 - y(t)$$
$$\dot{e}(t) = -\dot{y}(t)$$

Approximate Derivative Control

Effect of the Approximation (contd.)



Proportional-Derivative Control



Controller Structure

$$
\begin{aligned}
C(s) &= K\left(1 + T_D\frac{s}{s/\alpha + 1}\right)\\
&= K_P + K_D\frac{s}{s/\alpha + 1}
\end{aligned}
$$

- Tune K_P and K_D to get desired response
- Use Routh's table to determine range for stable values
- Derivative control increases damping reduces overshoot

PID Control

PID Control

Basic Idea

Controller Structure

$$C(s) = K_P + \frac{K_I}{s} + K_D \frac{s}{s/\alpha + 1}$$

- Tune K_P, K_I and K_D to get desired response
- Use Routh's table to determine range for stable values
- Proportional term **increases** ω_n and **decreases** ζ.
  - Improves rise time
  - Needs large gain to reduce steady-state error
- Integral term **increases** ω_n and **decreases** ζ.
  - Zero steady-state
  - May make the system unstable
- Derivative control **increases** damping – reduces overshoot and oscillations

PID Control

Example

System

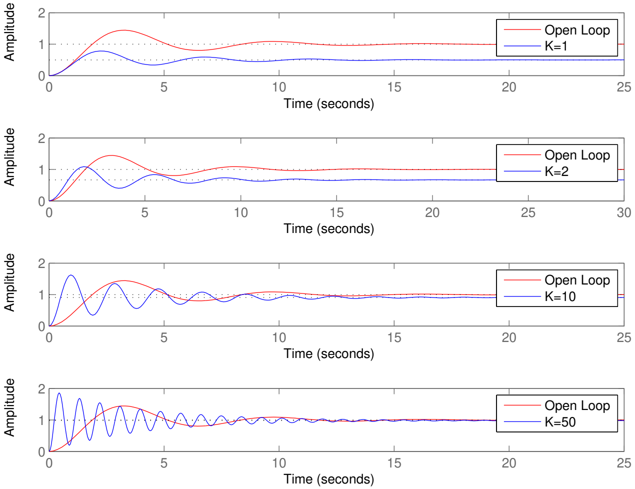
$$P(s) = \frac{1}{s^2 + 0.5s + 1}$$

Study behavior of this system with various control strategies.

- Proportional (P)
- Proportional Integral (PI)
- Proportional Derivative (PD)
- Proportional Integral and Derivative (PID)

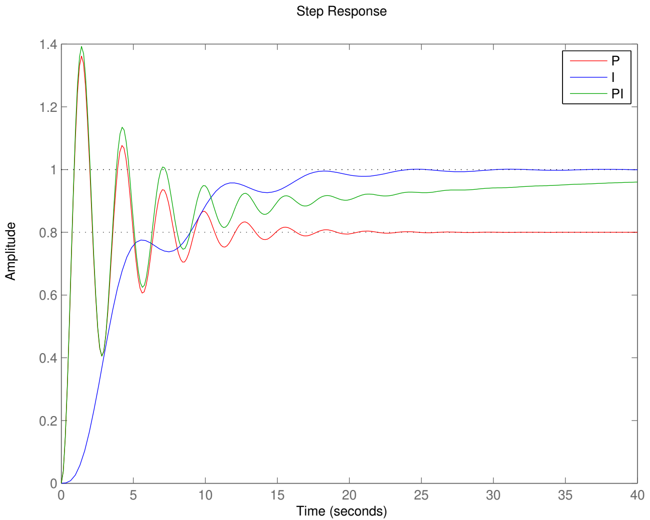
PID Control

Example: P



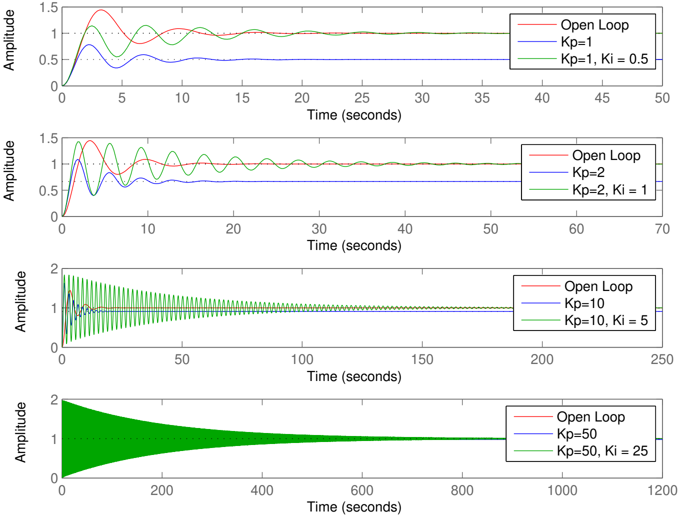
PID Control

Example: PI



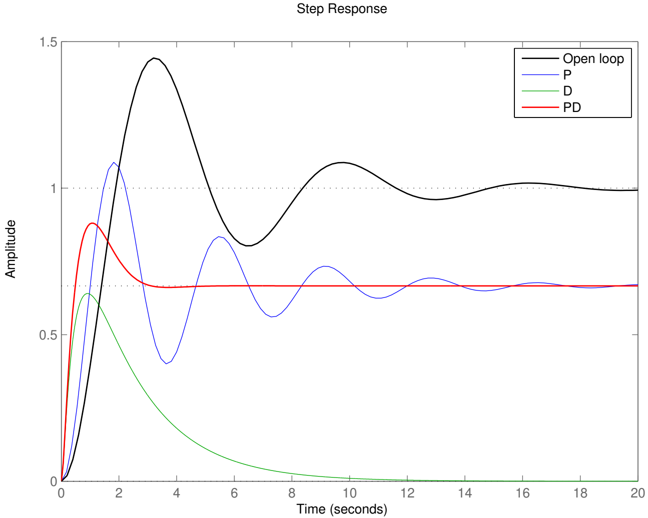
PID Control

Example: PI (contd.)



PID Control

Example: PD



PID Control

Example: PID